QUASIMARTINGALES WITH A LINEARLY ORDERED INDEX SET

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Abstract. We prove a version of Rao decomposition for quasi-martingales indexed by a linearly ordered set.

1. Introduction.

Rao's Theorem [12, Theorem 2.3, p. 89] asserts that, under the *usual conditions,* each quasi-martingale decomposes into the difference of two positive supermartingales. This decomposition turns out to be a crucial step in the proof of the Bichteler Dellacherie theorem, the fundamental theorem of semimartingale theory. We offer here a simple proof of this classical result without assuming right continuity neither for the filtration nor for the processes. Moreover, we allow the index set to be just a linearly ordered set. The setting is in fact the same as that proposed in [3], where a version of the Doob Meyer decomposition was obtained. A suitable example of a linearly ordered index set would be the whole real line.

In our general model the main source of difficulty (and of interest) lies in the need to forsake the stopping machinery, a most useful tool in stochastic analysis. We rather exploit the measure theoretic approach to stochastic processes, inaugurated by Doléans-Dade [5] and followed by many others, including Metivier and Pellaumail [9] and Dellacherie and Meyer [4]. In fact, we argue, the measure representation of processes is useful also in the absence of countable additivity, on which the literature has focused hitherto. The main result of this paper, Theorem 1, establishes that quasimartingales are isometrically isomorphic to locally countably additive measures. We show that all classical decompositions follow straightforwardly.

2. The Model.

The following notation will be convenient. $(\Omega, \mathscr{F}, P)$ will be a standard probability space, Δ a linearly ordered set and $(\mathscr{F}_\delta : \delta \in \Delta)$ an increasing family of sub σ algebras of $\mathscr{F}$. All σ algebras considered include the corresponding family of P null sets. $L^p(\mathscr{F}_\delta)$ will be preferred to $L^p(\Omega, \mathscr{F}_\delta, P)$, $\mathfrak{B}(\mathscr{F}_\delta)$ to $\mathfrak{B}(\Omega, \mathscr{F}_\delta)$ (the space of bounded, $\mathscr{F}_\delta$ measurable functions on Ω) and $\bar{\Omega}$ to $\Omega \times \Delta$. $\bar{\mathscr{F}}$ will be the augmentation of $\mathscr{F} \otimes 2^\Delta$ with respect to sets with P null projection on Ω. We shall also write $]\delta_1, \delta_2]$ for the (possibly empty) set $\{\delta \in \Delta : \delta_1 < \delta \leq \delta_2\}$.

All stochastic processes $(X_\delta : \delta \in \Delta)$ to be mentioned will be adapted, i.e. such that X_δ is $\mathscr{F}_\delta$ measurable for each $\delta \in \Delta$, but no form of right continuity is assumed. Two processes X and Y are considered as equal up to modification whenever $P(X_\delta = Y_\delta) = 1$ for all $\delta \in \Delta$ and all decompositions introduced later should be understood to be unique in the above sense.

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Of special importance is the collection $\mathscr{D}$ of finite subsets of Δ, with generic element $d = \{\delta_1 \le \ldots \le \delta_{N+1}\}$, a directed with respect to inclusion. $\mathscr{D}_\delta$ (resp. $\mathscr{D}_\delta^{\delta'}$) will denote the family of those sets $\{\delta_1 \le \ldots \le \delta_{N+1}\} \in \mathscr{D}$ such that $\delta_1 = \delta$ (resp. $\delta_1 = \delta$ and $\delta_{N+1} = \delta'$).

To each $d \in \mathscr{D}$ we associate the collection

$$\mathscr{P}^d = \left\{ \bigcup_{n=1}^{N} F_n \times]\delta_n, \delta_{n+1}] : F_n \in \mathscr{F}_{\delta_n},\ n = 1, \ldots, N \right\} \tag{2.1}$$

and define $\mathscr{P} = \bigcup_{d \in \mathscr{D}} \mathscr{P}^d$ and $\mathscr{E} = \bigcup_{d \in \mathscr{D}} \mathfrak{B}(\mathscr{P}^d)$. Abusing notation, we use the symbol $\mathscr{P}^d$ also for the operator $\mathscr{P}^d : \mathfrak{B}(\bar{\mathscr{F}}) \to \mathfrak{B}(\mathscr{P}^d)$ defined implicitly as

$$\mathscr{P}^d(U) = \sum_{n=1}^{N} P(U_{\delta_{n+1}} | \mathscr{F}_{\delta_n}) \mathbf{1}_{]\delta_n, \delta_{n+1}]} \tag{2.2}$$

A process A is increasing if $P(0 \le A_{\delta_1} \le A_{\delta_2}) = 1$ for $\delta_1 \le \delta_2$ and $\inf_{\delta \in \Delta} P(A_\delta) = 0$; it is integrable if $\sup_{\delta \in \Delta} P(|A_\delta|) < \infty$. An increasing process A is natural if

$$P\left(b \int f dA\right) = \operatorname*{LIM}_{d \in \mathscr{D}} P \int \mathscr{P}^d(b) f dA \qquad b \in L^\infty(\mathscr{F}),\ f \in \mathscr{E} \tag{2.3}$$

where LIM denotes here the Banach limit operator.

3. Quasi-martingales.

For each $d \in \mathscr{D}$ the d-variation of a process X is defined to be

$$V^d(X) = \sum_{n=1}^{N} \left| X_{\delta_n} - P(X_{\delta_{n+1}} | \mathscr{F}_{\delta_n}) \right| \quad \text{where} \quad d = \{\delta_1, \ldots, \delta_{N+1}\} \tag{3.1}$$

A process X is a quasi-martingale if

$$\|X\|_{\mathcal{Q}} = \sup \left\{ P\left(V^d(X)\right) : d \in \mathscr{D} \right\} < \infty \tag{3.2}$$

Quasi-martingales have been introduced and studied by Fisk [7], Orey [10] and Rao [12] who proved the classical decomposition theorem in its general form. Related results were obtained by Stricker [13] and [14] who proved uniqueness of the Rao decomposition. Our definition follows Rao [12] and differs from the one adopted by Stricker [14, p. 55] and by Dellacherie and Meyer [4, p. 98], which requires quasi-martingales to be integrable, a property that we do not impose here. However, in our setting supermartingales need not be quasi-martingales if not integrable.

Quasi martingales form a normed space with respect to $\|\cdot\|_{\mathcal{Q}}$ if we only identify processes which differ by a martingale. We will denote such space as $\mathcal{Q}$. A quasi-potential X is a quasi-martingale which admits a sequence $\langle \delta_n \rangle_{n \in \mathbb{N}}$ in Δ such that $\lim_n P(|X_{\bar{\delta}_n}|) = 0$ for any sequence $\langle \bar{\delta}_n \rangle_{n \in \mathbb{N}}$ such that $\bar{\delta}_n \ge \delta_n$ for $n = 1, 2, \ldots$. A potential is at the same time a quasi-potential and a positive supermartingale. Of course the difference of two potentials is a quasi-potential.

We shall make use of the following inequality, where $d' = \{\delta_1, \ldots, \delta_{N+1}\}$ and $d = \{\delta_1, \delta_{N+1}\}$

$$
\begin{aligned}
P\left(V^{d'}(X)\Big|\mathscr{F}_{\delta_1}\right) &= P\left(\sum_{n=1}^{N}\left|X_{\delta_n} - P(X_{\delta_{n+1}}|\mathscr{F}_{\delta_n})\right|\Bigg|\mathscr{F}_{\delta_1}\right)\\
&\geq \sum_{n=1}^{N}\left|P(X_{\delta_n}|\mathscr{F}_{\delta_1}) - P(X_{\delta_{n+1}}|\mathscr{F}_{\delta_1})\right|\\
&\geq \left|\sum_{n=1}^{N} P\left(X_{\delta_n} - X_{\delta_{n+1}}\big|\mathscr{F}_{\delta_1}\right)\right| \qquad (3.3)\\
&= \left|X_{\delta_1} - P\left(X_{\delta_{N+1}}\big|\mathscr{F}_{\delta_1}\right)\right|\\
&= V^d(X)
\end{aligned}
$$

We draw from (3.3) the following implications

Lemma 1. *Let X be a quasi-martingale and $\langle\delta_n\rangle_{n\in\mathbb{N}}$ an increasing sequence. The net $\left\langle P(V^d(X)|\mathscr{F}_\delta)\right\rangle_{d\in\mathscr{D}_\delta}$ and the sequence $\langle P(X_{\delta_n\vee\delta}|\mathscr{F}_\delta)\rangle_{n\in\mathbb{N}}$ both converge in $L^1(\mathscr{F}_\delta)$.*

Proof. Let $d' = \{\delta_{n_1}, \ldots, \delta_{n_{K+1}}\}$, $d = \{\delta_1, \ldots, \delta_{N+1}\} \in \mathscr{D}_\delta$ and $d \leq d'$. Then

$$V^{d'}(X) = \sum_{k=1}^{K}\left|X_{\delta_k} - P(X_{\delta_{k+1}}|\mathscr{F}_{\delta_k})\right| = \sum_{n=1}^{N}\sum_{\{\delta_n\leq\delta_{n_k}\leq\delta_{n+1}\}}\left|X_{\delta_{n_k}} - P(X_{\delta_{n_{k+1}}}|\mathscr{F}_{\delta_{n_k}})\right|$$

so that, by (3.3),

$$
\begin{aligned}
P\left(V^{d'}(X)\Big|\mathscr{F}_\delta\right) &= \sum_{n=1}^{N} P\left(P\left(\sum_{\{\delta_n\leq\delta_{n_k}\leq\delta_{n+1}\}}\left|X_{\delta_{n_k}} - P(X_{\delta_{n_{k+1}}}|\mathscr{F}_{\delta_{n_k}})\right|\Bigg|\mathscr{F}_{\delta_n}\right)\Bigg|\mathscr{F}_\delta\right)\\
&\geq \sum_{n=1}^{N} P\left(\left|X_{\delta_n} - P(X_{\delta_{n+1}}|\mathscr{F}_{\delta_n})\right|\big|\mathscr{F}_\delta\right)\\
&= P\left(V^d(X)\big|\mathscr{F}_\delta\right)
\end{aligned}
$$

The net $\left\langle P(V^d(X)|\mathscr{F}_\delta)\right\rangle_{d\in\mathscr{D}_\delta}$ is thus increasing but $P(V^d(X)) \leq \|X\|_{\mathcal{Q}}$. Convergence in L^1 follows then from [3, Lemma 1]. Again by (3.3), we get that $P\left(\sum_{n=1}^{N}\left|P(X_{\delta_n}|\mathscr{F}_\delta) - P(X_{\delta_{n+1}}|\mathscr{F}_\delta)\right|\right) \leq P(V^d(X)) \leq \|X\|_{\mathcal{Q}}$ which proves the second claim. □

4. A Characterisation.

Quasi-martingales can be characterised as elements of the space $ba(\bar{\mathscr{F}})$, i.e. as bounded finitely additive measures on $\bar{\mathscr{F}}$. $x \in ba(\bar{\mathscr{F}})$ is locally countably additive, in symbols $x \in \mathscr{M}^{loc}$, if x is countably additive in restriction to $\mathscr{P}^d$ for each $d \in \mathscr{D}$.

Theorem 1. *There is an isometric isomorphism between $\mathcal{Q}$ and $\mathscr{M}^{loc}$ determined by the identity*

$$x(f) = -\operatorname*{LIM}_{d\in\mathscr{D}} P\int \mathscr{P}^d(f)dX \qquad f \in \mathfrak{B}(\bar{\mathscr{F}}) \tag{4.1}$$

In particular,

(i) each $x \in \mathscr{M}^{loc}$ corresponds via (4.1) to one and only one integrable quasi-potential X;

(ii) $x \in \mathscr{M}^{loc}_+$ if and only if it corresponds via (4.1) to an integrable potential.

Remark 1. *If x and X are as in (4.1) then necessarily $x(f) = \mathrm{LIM}_{d\in\mathscr{D}}\, x(\mathscr{P}^d(f))$. Thus, if $|x|$ denotes the total variation measure of x and if $H \in \mathscr{P}$,*

$$|x|(H) = \sup_{\{g\in\mathfrak{B}(\bar{\mathscr{F}}),|g|\le 1\}} x(\mathbf{1}_H g) = \sup_{\{g\in\mathfrak{B}(\bar{\mathscr{F}}),|g|\le 1\}} \mathrm{LIM}_{d\in\mathscr{D}}\, x(\mathscr{P}^d(g)\mathbf{1}_H) \le \sup_{\{h\in\mathscr{E},|h|\le 1\}} x(h\mathbf{1}_H)$$

In other words, the restriction of $|x|$ to $\mathscr{P}$ coincides with total variation of $x|\mathscr{P}$.

Proof. Let X be a quasi-martingale. In view of the inequality $\left|P\int \mathscr{P}^d(f)dX\right| \le \|f\|_{\mathfrak{B}(\bar{\mathscr{F}})}\|X\|_{\mathcal{Q}}$, the right-hand side of (4.1) defines a continuous linear functional on $\mathfrak{B}(\bar{\mathscr{F}})$ and thus admits the representation as an integral with respect to some $x \in ba(\bar{\mathscr{F}})$, [6, corollary IV.5.3, p. 259] with $\|x\| \le \|X\|_{\mathcal{Q}}$. The correspondence between X and x is linear by the properties of the Banach limit. Given that $P\int hdX \ge \|X\|_{\mathcal{Q}} - \epsilon$ for all $\epsilon > 0$ and some $h \in \mathscr{E}$ with $\|h\| \le 1$, then $\|x\| = \|X\|_{\mathcal{Q}}$. Fix $d \in \mathscr{D}$ and let $\langle H_k\rangle_{k\in\mathbb{N}}$ be a decreasing sequence in $\mathscr{P}^d$ with empty intersection. With no loss of generality we may assume that $d = \{\delta, \delta'\}$. By definition, $H_k = F_k\times]\delta,\delta']$ for some $F_k \in \mathscr{F}_\delta$. Then, by Remark 1

$$\begin{aligned}
|x|(H_k) &= \sup_{\{h\in\mathscr{E},|h|\le 1\}} x(h\mathbf{1}_{H_k})\\
&\le \sup_{d\in\mathscr{D}_\delta^{\delta'}} P\left(F_k\sum_{n=1}^N |X_{\delta_n} - P(X_{\delta_{n+1}}|\mathscr{F}_{\delta_n})|\right)\\
&= \sup_{d\in\mathscr{D}_\delta^{\delta'}} P\left(F_k P(V^d(X)|\mathscr{F}_\delta)\right)
\end{aligned}$$

By Lemma 1(i), $P(V^d(X)|\mathscr{F}_\delta)$ converges in $L^1(\mathscr{F}_\delta)$ and therefore $P(F_kV^d(X))$ converges with $d \in \mathscr{D}_\delta^{\delta'}$ uniformly in k so that $\lim_k |x|(H_k) \le \lim_k \lim_{d\in\mathscr{D}_\delta^{\delta'}} P(F_kV^d(X)) = \lim_{d\in\mathscr{D}_\delta^{\delta'}} \lim_k P(F_kV^d(X)) = 0$ which proves that $|x| \in \mathscr{M}^{loc}$ and, *a fortiori*, that the same is true of x. If X is an integrable potential, it is then a quasi-martingale associated to some $x \in \mathscr{M}_+^{loc}$.

Fix now $x \in \mathscr{M}^{loc}$ and let $\langle\delta_x(n)\rangle_{n\in\mathbb{N}}$ and $\langle\delta^x(n)\rangle_{n\in\mathbb{N}}$ be monotonic sequences in Δ such that

$$\lim_n |x|(]\delta_x(n), \delta^x(n)]) = \sup_{\delta,\delta'\in\Delta} |x|(]\delta,\delta']) \tag{4.2}$$

Define $x^\delta,\ x_\delta \in ba(\mathscr{F})$ implicitly as

$$x^\delta(F) = \lim_n x(F\times]\delta_x(n),\delta]), \quad x_\delta(F) = \lim_n x(F\times]\delta,\delta^x(n)]) \qquad F \in \mathscr{F} \tag{4.3}$$

Given that x is locally countably additive, then $x_\delta|\mathscr{F}_\delta \ll P|\mathscr{F}_\delta$ and we denote by X_δ the corresponding Radon Nykodim derivative. Clearly, $P(|X_\delta|) \le \|x\|$. If $F \in \mathscr{F}_\delta$ and $\delta' \ge \delta$, then $P(\mathbf{1}_F(X_\delta - P(X_{\delta'}|\mathscr{F}_\delta)) = x(F\times]\delta,\delta'])$ so that (4.1) is satisfied. Moreover $\lim_n P(|X_{\bar\delta_n}|) \le \lim_n \lim_k |x|(]\delta^x(n), \delta^x(k)]) = 0$ for any sequence $\langle\bar\delta_n\rangle_{n\in\mathbb{N}}$ such that $\bar\delta_n \ge \delta^x(n)$ for all n. If $x \ge 0$ then it is obvious that X is a positive super-martingale. If Y were another integrable quasi-potential corresponding to x via (4.1) then $X - Y$ would be at the same time a quasi-potential and a martingale. But then, for all $\delta \in \Delta$ and some sequence $\langle\bar\delta_n\rangle_{n\in\mathbb{N}}$, $P(|X_\delta - Y_\delta|) \le \lim_n P(|X_{\bar\delta_n\vee\delta} - Y_{\bar\delta_n\vee\delta}|) = 0$. □

A conclusion implicit in Theorem 1 is that all quasi-potentials are integrable. Another consequence is the following version of a result of Stricker [14, Theorem 1.2, p. 55].

Corollary 1 (Stricker)**.** *Let X be a quasi-martingale, $(\mathscr{G}_\delta : \delta \in \Delta)$ a sub filtration of $(\mathscr{F}_\delta : \delta \in \Delta)$ and define $X_\delta^{\mathscr{G}} = P(X_\delta|\mathscr{G}_\delta)$. Then the process $X^{\mathscr{G}}$ is itself a quasi-martingale.*

Proof. The restriction to the subfiltration preserves local countable additivity of the corresponding measure. □

The issue of the invariance of the process structure with respect to a change of the underlying filtration was also addressed in [2].

5. Decompositions.

The following version of Riesz decomposition follows immediately from Theorem 1.

Corollary 2. *A process X is a quasi-martingale if and only if it decomposes into the sum*

$$X = M + B \tag{5.1}$$

of a martingale M and a quasi-potential B. The decomposition is then unique. Moreover, if X, M and B are as in (5.1), then

(*i*) *X is a (positive) supermartingale if and only if B is a potential (and M is positive);*

(*ii*) *If for any δ there is an integer n such that $\delta^x(n) \geq \delta$, then M is uniformly integrable if and only if $\{X_{\delta^x(n)} : n = 1, 2, \dots\}$ is so.*

Proof. By Theorem 1, X corresponds to some $x \in \mathscr{M}^{loc}$ and x, in turn, to a unique quasi-potential, B. Given that $\|X - B\|_{\mathcal{Q}} = 0$, then $M = X - B$ is indeed a martingale. To be more explicit, we write

$$X_\delta = \lim_n P\left(X_{\delta^x(n)\vee\delta} \middle| \mathscr{F}_\delta\right) + \left.\frac{dx_\delta}{dP}\right|_{\mathscr{F}_\delta} = M_\delta + B_\delta \tag{5.2}$$

the sequence $\langle \delta^x(n) \rangle_{n\in\mathbb{N}}$ being defined as in (4.2). To see that (5.2) is well defined, observe that the limit appearing in it exists in $L^1(\mathscr{F}_\delta)$ (by Lemma 1) and that $x_\delta(F) = \lim_n x(F \times]\delta, \delta^x(n)]) = \lim_n x(F \times]\delta, \delta^x(n) \vee \delta]) = \lim_n P(\mathbf{1}_F(X_\delta - X_{\delta^x(n)\vee\delta}))$ for all $F \in \mathscr{F}_\delta$. Moreover,

$$|P(\mathbf{1}_F(M_\delta - M_{\delta'}))| = \lim_n \left|P(\mathbf{1}_F(X_{\delta^x(n)\vee\delta} - X_{\delta^x(n)\vee\delta'}))\right| \leq |x|\left((]\delta^x(n), \delta^x(n) \vee \delta'])\right) = 0$$

so that M is a martingale and it is positive if $X \geq 0$.

Claim (i) is a consequence of the fact that $x \geq 0$ whenever X is a supermartingale. If $\{X_{\delta^x(n)} : n = 1, 2, \dots\}$ is uniformly integrable then $P(|M_\delta|\mathbf{1}_F) \leq \sup_n P(|X_{\delta^x(n)}|\mathbf{1}_F)$ for any $F \in \mathscr{F}_\delta$, which implies uniform integrability of M. The converse is obvious given that the collection $\{M_{\delta^x(n)} : n = 1, 2, \dots\}$ is uniformly integrable by assumption while $\lim_n P(|B_{\delta^x(n)}|) = 0$ by construction. □

It is noteworthy that, when the index set is order dense, e.g. $\Delta = \mathbb{R}$, uniform integrability of the martingale M does not require more than uniform integrability of X *along a given sequence*, i.e. of a countable set. In [3] it was shown that the class D property could likewise be restricted to consider only stopping times with countably many values.

Corollary 3 (Rao and Stricker)**.** *A process X is a quasi-potential (resp. an integrable quasi-martingale) if and only if it decomposes into the difference*

$$X = X' - X'' \tag{5.3}$$

of two potentials (resp. positive, integrable supermartingales) such that $\|X\|_{\mathcal{Q}} = \|X'\|_{\mathcal{Q}} + \|X''\|_{\mathcal{Q}}$. Any other decomposition $Y' - Y''$ of X as the difference of two potentials (resp. positive, integrable supermartingales) is such that $Y' - X'$ and $Y'' - X''$ are potentials (resp. positive, integrable, supermartingales).

Proof. Let X be a quasi-martingale isomorphic to x, $x' - x''$ be its Jordan decomposition and X' and X'' the associated potentials as of Theorem 1.(i). Then, $\|X\|_{\mathcal{Q}} = \|x\| = \|x'\| + \|x''\| = \|X'\| + \|X''\|$. If X is a quasi potential, then (5.3) follows from (5.2). If X is an integrable quasi-martingale with Riesz decomposition $M + B$, then M is integrable. Define

$$M'_\delta = \lim_n P\left(M^+_{\delta^x(n)\vee\delta}\Big|\mathscr{F}_\delta\right) \quad \text{and} \quad M''_\delta = \lim_n P\left(M^-_{\delta^x(n)\vee\delta}\Big|\mathscr{F}_\delta\right)$$

Observe that convergence takes place in $L^1(\mathscr{F}_\delta)$ as the corresponding sequences are increasing but norm bounded. Let $B = B' - B''$ be the decomposition of the quasi-potential B into the difference of two potentials, following from the previous step and set $X' = M' + B'$ and $X'' = M'' + B''$. Clearly, X' and X'' are positive supermartingales and satisfy (5.3). Moreover $\|X\|_{\mathcal{Q}} = \|B'\|_{\mathcal{Q}} + \|B''\|_{\mathcal{Q}} = \|X'\|_{\mathcal{Q}} + \|X''\|_{\mathcal{Q}}$. Let $Y' - Y''$ be a decomposition of X into two potentials (resp. positive, integrable supermartingales). Then Y' corresponds to some $y' \in \mathscr{M}^{loc}$ such that necessarily $y' \geq x'$ so that the potential part of Y' exceeds the corresponding component of X'. To conclude that the same is true of the martingale part (if any), observe that $Y' \geq X^+$ so that

$$\begin{aligned}
\lim_n P\left(Y'_{\delta^x(n)\vee\delta}\Big|\mathscr{F}_\delta\right) &\geq \lim_n P\left(X^+_{\delta^x(n)\vee\delta}\Big|\mathscr{F}_\delta\right)\\
&\geq \lim_k \lim_n P\left(P\left(X_{\delta^x(n)\vee\delta}\big|\mathscr{F}_{\delta^x(k)\vee\delta}\right)^+\Big|\mathscr{F}_\delta\right)\\
&= \lim_k P\left(M'_{\delta^x(k)\vee\delta}\Big|\mathscr{F}_\delta\right)\\
&= M'_\delta
\end{aligned}$$

We then obtain from (5.2) that $Y' - X'$ is indeed a potential (resp. positive, integrable supermartingale). It is easily seen that the same is true of Y''. □

Local countable additivity has not been much studied in the literature, given the exclusive attention paid to the class $ca(\mathscr{P})$ in the literature. Such attention was motivated by the proof offered by Doléans-Dade [5] that the Doob Meyer decomposition is equivalent, *under the usual conditions*, to countable additivity over the predictable σ algebra. It was Mertens [8] the first to provide a version of this result without invoking such regularity assumptions, a result later proved in greater generality by Dellacherie and Meyer [4, Theorem 20, p. 414]. A proof, in the setting of linearly ordered index set, was given in [3] based on a suitable extension of the *class D property*. A purely measure theoretic proof was provided in [2, theorem 4, p. 597] for processes indexed by the positive reals. This same argument will now be easily adapted to the general setting considered here.

Theorem 2. *Let $x \in \mathscr{M}^{loc}_+$ be isomorphic to a potential X and define $x_{\mathscr{F}}$ implicitly as*

$$x_{\mathscr{F}}(F) = x(F \times \Delta) \qquad F \in \mathscr{F} \tag{5.4}$$

Then $x_{\mathscr{F}} \ll P$ if and only if there exist $M \in L^1_+$ and an increasing, integrable, natural process A such that

$$X_\delta = P(M|\mathscr{F}_\delta) - A_\delta \qquad P \text{ a.s.},\ \delta \in \Delta \tag{5.5}$$

The decomposition (5.5) is unique.

Proof. Assume that $x_{\mathscr{F}} \ll P$ and recall the definition (4.3) of $x^\delta \in ba(\mathscr{F})_+$. The inequality $x^\delta \leq x_{\mathscr{F}}$ guarantees that $x^\delta \ll P$ for all $\delta \in \Delta$. Let A_δ be the corresponding Radon Nykodim derivative. Then

$P(0 \le A_\delta \le A_{\delta'}) = 1$ for all $\delta \le \delta'$. Choose $b \in L^\infty(\mathscr{F})$ and $f \in \mathscr{E}$. Then $f = f\mathbf{1}_{]\delta',\delta]}$ for some $\delta', \delta \in \Delta$ and (4.1) implies

$$\begin{aligned} P\left(b\int f dA\right) &= x^\delta(bf) \\ &= x(bf\mathbf{1}_{]\delta',\delta]}) \\ &= \operatorname*{LIM}_{d\in\mathscr{D}} x\left(\mathscr{P}^d(b)f\mathbf{1}_{]\delta',\delta]}\right) \\ &= \operatorname*{LIM}_{d\in\mathscr{D}} x^\delta\left(\mathscr{P}^d(b)f\right) \\ &= \operatorname*{LIM}_{d\in\mathscr{D}} P\int \mathscr{P}^d(b)f dA \end{aligned}$$

so that A is natural (and thus adapted). Moreover, letting $M = dx_{\mathscr{F}}/dP$ and $F \in \mathscr{F}_\delta$ we have

$$P(\mathbf{1}_F X_\delta) = \lim_n x(F\times]\delta, \delta^x(n)\vee\delta]) + \lim_n P(\mathbf{1}_F X_{\delta^x(n)\vee\delta}) = x_\delta(F) = x_{\mathscr{F}}(F) - x^\delta(F) = P(\mathbf{1}_F(M - A_\delta))$$

Suppose that $N - B$ is another such decomposition. Then A and B are both natural and then for each $F \in \mathscr{F}_\delta$,

$$P(\mathbf{1}_F A_\delta) = \operatorname*{LIM}_{d\in\mathscr{D}} \lim_n P\int_{\delta_x(n)}^\delta \mathscr{P}^d(F)dA = \operatorname*{LIM}_{d\in\mathscr{D}} \lim_n P\int_{\delta_x(n)}^\delta \mathscr{P}^d(F)dB = P(\mathbf{1}_F B_\delta)$$

□

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